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Extra Derivative Implicit Block Methods for Integrating General Second Order Initial Value Problems

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ABSTRACT

This paper focuses on the construction of two-point and three-point implicit block methods for solving general second order Initial Value Problems. The proposed methods are formulated using Hermite Interpolating Polynomial. The block methods approximate the numerical solutions at more than one point at a time directly without reducing the equation into the first order system of ordinary differential equations. In the derivation of the method, the higher derivative of the problem is incorporated into the formula to enhance the efficiency of the proposed methods. The order and zero- stability of the methods are also presented. Numerical results presented show the efficiency of these methods compared to the existing block methods.

Keywords: Block methods, extra derivative, second order IVPs

INTRODUCTION

Many researchers have focused on the block method for directly solving general second order initial value problems (IVPs), whereby the IVPs are not reduced to system of first order IVPs. Awoyemi et al. (2011) used the collocation technique to develop block linear multistep methods to solve second order IVPs. Majid et al. (2012) used two-point block method to solve general second order IVPs. Badmus (2014) developed an efficient seven-point hybrid block method for the direct solution of general second order IVPs. Abdelrahim and Omar (2016) developed a single-step hybrid block method of order five, for directly solving second order ordinary differential equations (ODEs). For

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solving the same type of problems, Ramos et al. (2016), developed an efficient Falkner-type method of order two and three. Waeleh and Majid (2017) derived block method to solve second order IVPs using variable stepsize code. Nasir et al. (2018) presented the diagonal block method of order four, for solving the second-order boundary value problems with Robin boundary conditions. While Singh and Ramos (2019) derived an optimized two-step hybrid block method which was formulated in variable step-size mode for integrating the general second order IVPs directly.

In most of the previously mentioned work, the methods did not have the extra derivative in the formulation of the methods. The aim of having the extra derivative in the formulation of the method is that, numerical solutions which are very accurate can be obtained. Furthermore, most of the block methods in the literature were derived using collocation and interpolation technique and some of them were derived using linear operator, which require more computational effort.

In this paper, we derived the methods using integration technique which was much simpler than the collocation and linear operator techniques. Previous work on block method which were derived using integration technique, only used Newton interpolation for the function on the right-hand side of the integration. In this research the function was replaced by Hermite interpolation, so that the extra derivatives of the problems to be solved could be included into the formula. Here, block methods with extra derivative are derived for directly solving the general second order ODEs (Equation 1).

$$y'' = f(t, y, y'), \qquad y(a) = y_0, y'(a) = y'_0 \quad a \le t \le b \tag{1}$$

The first derivative of f with respect to t can be written as

$$y''' = f'(t, y, y') = f_t + y' f_y + f f_{y'} = g(t, y, y').$$

Hermite Interpolating Polynomial P, can be defined by Equation 2:

$$P(t) = \sum_{i=0}^{n} \sum_{k=0}^{m_{i-1}} f_i^{(k)} L_{i,k}(t), \tag{2}$$

where $f_i = f(t_i), t_i = a + ih,\ i = 0,1,\ldots,n$ and $h = \frac{b-a}{n}$, n is a positive integer. $L_{i,k}(t)$ is the generalized Lagrange polynomial which can be defined by

$$L_{i,m_i}(t) = \ell_{i,m_i}(t), i = 0,1,\ldots,n,$$

$$\ell_{i,k}(t) = \frac{(t-t_i)^k}{k!} \prod_{j=0, j\neq i}^{n} (\frac{t-t_j}{t_i-t_j})^m j, i = 0,1,\ldots,n, k = 0,1,\ldots,m_i.$$

And recursively for $k = m_i - 2, m_i - 3, \ldots, 0$.

$$L_{i,k}(t) = \ell_{i,k}(t) - \sum_{v=k+1}^{m_i-1} \ell_{i,k}^{(v)}(t_i) L_{i,v}(t).$$

MATERIALS AND METHODS

Derivation of the Methods

In two-point block method, the interval $[a,b]$ contains two points for each block. To evaluate the first point, y_{n+1} and y'_{n+1} at t_{n+1}, we integrate (1) once and twice over the interval $[t_n, t_{n+1}]$, which gives Equation 3

$$\int_{t_n}^{t_{n+1}} y''\, dt = \int_{t_n}^{t_{n+1}} f(t, y, y') dt. \quad (3)$$

and Equation 4

$$\int_{t_n}^{t_{n+1}} \int_{t_n}^{t} y'\, dt\, dt = \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} f(t, y, y') dt\, dt. \quad (4)$$

Let $t_{n+1} = t_n + h$ and substituting into Equation 3 and 4, we have Equation 5 and 6

$$y'(t_{n+1}) = y'(t_n) + \int_{t_n}^{t_{n+1}} f(t, y, y') dt, \quad (5)$$

$$y'(x_{n+1}) = y(t_{n+1}) = y(t_n) + h\, y'(t_n) + \int_{t_n}^{t_{n+1}} (t_{n+1} - t) f(t, y, y')\, dt. \quad (6)$$

Then, $f(t, y, y')$ in Equation 5 and 6 will be replaced by Hermite Interpolating Polynomial in Equation 2 which is defined by $P_2(t)$ as follows (Equation 7):

$$\begin{aligned} P_2(t) = \ & [(\frac{t - t_{n+1}}{t_n - t_{n+1}})^2 (\frac{t - t_{n+2}}{t_n - t_{n+2}})^2 + (\frac{2}{t_n - t_{n+1}})(\frac{2}{t_n - t_{n+2}})(t - t_n)(\frac{t - t_{n+1}}{t_n - t_{n+1}})^2 \\ & (\frac{t - t_{n+2}}{t_n - t_{n+2}})^2] f_0 + [(\frac{t - t_{n+1}}{t_n - t_{n+1}})^2 (\frac{t - t_{n+2}}{t_n - t_{n+2}})^2] f_1 + [(\frac{t - t_n}{t_{n+2} - t_n})^2 \\ & (\frac{t - t_{n+1}}{t_{n+2} - t_{n+1}})^2 + (\frac{2}{t_{n+2} - t_{n+1}})(\frac{2}{t_{n+2} - t_n})(t - t_{n+2})(\frac{t - t_n}{t_{n+2} - t_n})^2 \\ & (\frac{t - t_{n+1}}{t_{n+2} - t_{n+1}})^2] f_2 + [(t - t_n)(\frac{t - t_{n+1}}{t_n - t_{n+1}})^2 (\frac{t - t_{n+2}}{t - t_{n+2}})^2] g_0 + [(t - t_{n+1}) \\ & (\frac{t - t_n}{t_{n+1} - t_n})^2 (\frac{t - t_{n+2}}{t_{n+2} - t_n})^2] g_1 + [(t - t_{n+2})(\frac{t - t_n}{t_{n+2} - t_n})^2 (\frac{t - t_{n+1}}{t_{n+2} - t_{n+1}})^2] g_2 \end{aligned} \quad (7)$$

Where f_0, f_1 and f_2 are the function f (Equation 1) at the first, second and third point of the interpolation, while g_0, g_1 and g_2 are the derivatives respectively (Equation 8).

Let $t = t_{n+2} + s\,h$ and

$$s = \frac{t - t_{n+2}}{h}. \tag{8}$$

Taking $dt = h\,ds$ and change the limit of integration from –2 to –1 in Equation 5 and 6 we obtain Equation 9 and 10

$$y'(x_{n+1}) = y'(x_n) + \int_{-2}^{-1}[\,f_0L_{0,0}(s) + f_1L_{1,0}(s) + f_2L_{2,0}(s) + g_0L_{0,1}(s) + g_1L_{1,1}(s) + g_2L_{2,1}(s)]\,h\,ds, \tag{9}$$

$$y(x_{n+1}) = y(x_n) + hy'(x_n) + \int_{-2}^{-1}(-h - sh)\,[f_0L_{0,0}(s) + f_1L_{1,0}(s) + f_2L_{2,0}(s\Big) g_1L_{1,1}(s) + g_2L_{2,1}(s)]\,h\,ds\,. \tag{10}$$

where

$$L_{0,0}(s) = (\tfrac{s^2}{4}(s+1)^2 + \tfrac{3}{4}s^2(s+2)(s+1)^2),\; L_{1,0}(s) = s^2(s+2)^2,$$

$$L_{2,0}(s) = \tfrac{1}{4}((s+2)^2(s+1)^2 - 3s(s+2)^2(s+1)^2),$$

$$L_{0,1}(s) = h(\tfrac{s^2}{4}(s+2)(s+1)^2),\; L_{1,1}(s) = h(s^2(s+1)(s+2)^2),$$

$$L_{2,1}(s) = h(\tfrac{s}{4}(s+1)^2(s+2)^2).$$

Evaluating the integrals in Equation 9 and 10 produces the first formula of the two-point implicit block method as follows (Equation 11 and 12):

$$y'_{n+1} = y'_n + \frac{h}{240}[101f_n + 128f_{n+1} + 11f_n] + \frac{h^2}{240}[13g_n - 40g_{n+1} - 3g_{n+2}] \tag{11}$$

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{42}[13f_n + 7f_{n+1} + f_{n+2}\Big] + \frac{h^3}{1680}[59g_n - 128g_{n+1} - 11g_{n+2}] \tag{12}$$

Integrating Equation 1 once and twice over the $[t_{n+1}, t_{n+2}]$ to obtain the approximate solutions of y_{n+2} and y'_{n+2}, we have Equation 13

$$\int_{t_{n+1}}^{t_{n+2}} y''\,dt = \int_{t_{n+1}}^{t_{n+2}} f\,(t,y,y')dt. \tag{13}$$

and Equation 14

$$\int_{t_{n+1}}^{t_{n+2}} \int_{t_{n+1}}^{t} y'\;dt\,dt = \int_{t_{n+1}}^{t_{n+2}} \int_{t_{n+1}}^{t} f\,(t,y,y')dt\,dt. \tag{14}$$

Taking $t_{n+2} = t_{n+1} + h$ and substituting into Equation 13 and 14 we have Equation 15 and 16

$$y'(t_{n+2}) = y'(t_{n+1}) + \int_{t_{n+1}}^{t_{n+2}} f\,(t,y,y')dt \tag{15}$$

$$y(t_{n+2}) = y(t_{n+1}) + hy'(t_n) + \int_{t_{n+1}}^{t_{n+2}} (t_{n+2} - t) f\,(t,y,y')dt \tag{16}$$

Replaced $f(t,y,y')$ in Equation 15 and 16 by Hermite Interpolating Polynomial in Equation 7 and changing the limit of integration from –1 to 0 in Equation 15 and 16, we obtain Equation 17 and 18

$$\begin{aligned} y'(x_{n+2}) = y'(x_{n+1}) + \int_{-1}^{0} [\,& f_0 L_{0,0}(s) + f_1 L_{1,0}(s) + f_2 L_{2,0}(s) \\ & + g_0 L_{0,1}(s) + g_1 L_{1,1}(s) + g_2 L_{2,1}(s)]\;h\,ds. \end{aligned} \tag{17}$$

$$\begin{aligned} y(x_{n+2}) = y(x_{n+1}) + hy'(x_{n+1}) + \int_{-1}^{0} (-sh)[& f_0 L_{0,0}(s) + f_1 L_{1,0}(s) \\ & + f_2 L_{2,0}(s) + g_0 L_{0,1}(s) + g_1 L_{1,1}(s) + g_2 L_{2,1}(s)]\;h\,ds. \end{aligned} \tag{18}$$

Evaluating the integrals in Equation 17 and 18, produces the second formula of the two-point implicit block method as follows (Equation 19 and 20):

$$y'_{n+2} = y'_{n+1} + \frac{h}{240}[11f_n + 128f_{n+1} + 101f_{n+2}] + \frac{h^2}{240}[3g_n + 40g_{n+1} - 13g_{n+2}]. \tag{19}$$

$$y_{n+2} = y_{n+1} + hy'_{n+1} + \frac{h^2}{1680}[37f_n + 616f_{n+1} + 187f_{n+2}] + \frac{h^3}{80}[5g_n + 76g_{n+1} - 16g_{n+2}]. \tag{20}$$

We denote the formula as two-point second derivative block implicit method or 2PSDBI(2).

In the three-point block, each block contains three points. The values of y_{n+1}, y_{n+2} and y_{n+3} at the point t_{n+1}, t_{n+2} and $t_{n+3}1$ are calculated concurrently in a block. The approach is similar to the derivation of the two-point implicit method. Equation 1 will be integrated once and twice over the intervals $[t_n, t_{n+1}]$, $[t_{n+1}, t_{n+2}]$ and $[t_{n+2}, t_{n+3}]$ to obtain the approximate solutions of $y_{n+1}, y'_{n+1}, \; y_{n+2}, y'_{n+2}, y_{n+3}$ and y'_{n+3}. Define $P_3(t)$ as follows (Equation 21):

$$P_3(t) = [(\frac{t-t_{n+1}}{t_n - t_{n+1}})(\frac{t-t_{n+1}}{t_n - t_{n+1}})(\frac{t-t_{n+3}}{t_n - t_{n+3}})^2 + ((\frac{-1}{t_n - t_{n+1}}) + (\frac{-1}{t_n - t_{n+2}}) + (\frac{-2}{t_n - t_{n+3}}))$$

$$((t-t_n)(\frac{t-t_{n+1}}{t_n-t_{n+1}})(\frac{t-t_{n+2}}{t_n-t_{n+2}})(\frac{t-t_{n+3}}{t_n-t_{n+3}})^2)]f_0 + [(\frac{t-t_n}{t_{n+1}-t_n})^2(\frac{t-t_{n+2}}{t_{n+1}-t_{n+2}})(\frac{t-t_{n+3}}{t_{n+1}-t_{n+3}})^2]f_1 +$$

$$[(\frac{t-t_n}{t_{n+2}-t_n})^2(\frac{t-t_{n+1}}{t_{n+2}-t_{n+1}}) \quad (\frac{t-t_{n+3}}{t_{n+2}-t_{n+3}})^2]f_2 + [(\frac{t-t_n}{t_{n+3}-t_n})^2(\frac{t-t_{n+1}}{t_{n+3}-t_{n+1}})(\frac{t-t_{n+2}}{t_{n+3}-t_{n+2}}) - ((\frac{2}{t_{n+3}-t_n})$$

$$+(\frac{1}{t_{n+3}-t_{n+1}}) + (\frac{1}{t_{n+3}-t_{n+2}}))((t-t_{n+3})(\frac{t-t_n}{t_{n+3}-t_n})^2(\frac{t-t_{n+1}}{t_{n+3}-t_{n+1}}) \; (\frac{t-t_{n+2}}{t_{n+3}-t_{n+2}}))]f_3$$

$$+((t-t_n)(\frac{t-t_{n+1}}{t_n-t_{n+1}})(\frac{t-t_{n+2}}{t_n-t_{n+2}})(\frac{t-t_{n+3}}{t_n-t_{n+3}})^2)g_0 + ((t-t_{n+3})(\frac{t-t_n}{t_{n+3}-t_n})^2(\frac{t-t_{n+1}}{t_{n+3}-t_{n+1}})(\frac{t-t_{n+2}}{x_{n+3}-x_{n+2}}))g_3. \tag{21}$$

Hermite Interpolating Polynomial in Equation 21 will interpolate $f(x,y,y')$ and let $t = t_{n+3} + s\,h$ and $s = \frac{t-t_{n+3}}{h}$. For each evaluation of $y_{n+1}, y'_{n+1}, \; y_{n+2}, y'_{n+2}$ and y_{n+3}, y'_{n+3}, we obtained the formulae which can be written as follows (Equation 22, 23, 24, 25, 26 and 27):

$$y'_{n+1} = y'_n + \frac{h}{6480}[3463f_n + 3537f_{n+1} - 783f_{n+2} + 263f_{n+3}] + \frac{h^2}{1080}[97g_n - 17g_{n+3}], \tag{22}$$

$$y_{n+1} = y_n + h\,y'_n + \frac{h^2}{45360}[16384f_n + 7857f_{n+1} - 2376f_{n+2} + 815f_{n+3}] + \frac{h^3}{7560}[388g_n - 53g_{n+3}], \tag{23}$$

$$y'_{n+2} = y'_{n+1} + \frac{h}{80}[-7f_n + 47f_{n+1} + 47f_{n+2} - 7f_{n+3}] + \frac{h^2}{360}[-11g_n + 11g_{n+3}], \tag{24}$$

$$y_{n+2} = y_{n+1} + h\,y'_{n+1} + \frac{h^2}{15120}[-715f_n + 5832f_{n+1} + 3051f_{n+2} - 608f_{n+3}] + \frac{h^3}{2520}[-41g_n + 36g_{n+3}], \tag{25}$$

$$y'_{n+3} = y'_{n+2} + \frac{h}{6480}[263f_n - 783f_{n+1} + 3537f_{n+2} + 3463f_{n+3}] + \frac{h^2}{1080}[17g_n - 97g_{n+3}], \tag{26}$$

$$y_{n+3} = y_{n+2} + h\,y'_{n+2} + \frac{h^2}{1680}[38f_n - 115f_{n+1} + 626f_{n+2} + 291f_{n+3}] + \frac{h^3}{2520}[22g_n - 97g_{n+3}]. \tag{27}$$

This method is denoted as three-point second derivative block implicit method or 3PSDBI(2).

Order and Error Constant

The local truncation error associated with the normalized form of the proposed method can be defined as the linear difference operator (Equation 28)

$$L[\psi(t);h] = \sum_{i=0}^{k}[\alpha_i\psi(t+jh) - h\beta_i\psi'(t+jh) - h^2\gamma_i\psi''(t+jh) - h^3\delta_i\psi'''(t+jh)]. \tag{28}$$

Further detail can be seen in Fatunla (1995). Assuming that $\psi(t)$ is sufficiently differentiable, Equation 28 can be expanded as a Taylor series expansion about the point t to obtain the expression $L[\psi(t);h] = \mathbb{C}_0\psi(t) + \mathbb{C}_1 h\psi'(t) + \ldots + \mathbb{C}_p h^p\psi^{(p)}(t) + \ldots$, where the constant coefficients $\mathbb{C}_p, p = 0,1,\ldots$ are given as follows (Equation 29):

$$_0 = \sum_{i=0}^{k}\alpha_j, \quad \mathbb{C}_1 = \sum_{i=0}^{k} j\,\alpha_j - \sum_{i=0}^{k}\beta_j,$$

$$\vdots$$

$$\mathbb{C}_p = \frac{1}{p!}\sum_{i=0}^{k} j^p\alpha_j - \frac{1}{(p-1)!}\sum_{i=0}^{k} j^{p-1}\beta_j - \frac{1}{(p-2)!}\sum_{i=0}^{k} j^{p-2}\gamma_j - \frac{1}{(p-3)!}\sum_{i=0}^{k} j^{p-3}\delta_j, p = 3,4 \tag{29}$$

It can be said that the proposed method has order p if $\mathbb{C}_0 = \mathbb{C}_1 = \ldots \mathbb{C}_p = \mathbb{C}_{p+1} = 0, \mathbb{C}_{p+2} \neq 0$. Therefore, $\mathbb{C}_{p+2}$ is the error constant and $\mathbb{C}_{p+2}h^{p+2}\psi^{(p+2)}(t_n)$ is the principal local truncation error at the point t_n.

The formulae of the two-point implicit block method given by Equation 11, 12, 19 and 20 can be written in the form of a matrix as follows:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} y_{n-1} \\ y_n \\ y_{n+1} \\ y_{n+2} \end{bmatrix} = h\begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} y'_{n-1} \\ y'_n \\ y'_{n+1} \\ y'_{n+2} \end{bmatrix} +$$

$$h^2\begin{bmatrix}0 & \frac{101}{240} & \frac{128}{240} & \frac{11}{240} \\ 0 & \frac{18}{42} & \frac{7}{42} & \frac{1}{42} \\ 0 & \frac{11}{240} & \frac{128}{240} & \frac{101}{240} \\ 0 & \frac{37}{1680} & \frac{616}{1680} & \frac{187}{1680}\end{bmatrix}\begin{bmatrix}f_{n-1}\\ f_n\\ f_{n+1}\\ f_{n+2}\end{bmatrix} + h^3\begin{bmatrix}0 & \frac{13}{240} & \frac{-40}{240} & \frac{-3}{240} \\ 0 & \frac{59}{1680} & \frac{-128}{1680} & \frac{-11}{1680} \\ 0 & \frac{3}{240} & \frac{40}{240} & \frac{-13}{240} \\ 0 & \frac{5}{840} & \frac{76}{840} & \frac{-16}{840}\end{bmatrix}\begin{bmatrix}g_{n-1}\\ g_n\\ g_{n+1}\\ g_{n+2}\end{bmatrix}$$

Or Equation 30

$$\alpha Y_m = h\beta Y'_m + h^2\gamma F_m + h^3\delta G_m \tag{30}$$

where α, β, γ and δ are the (4x4) matrix coefficients of Y_m, Y'_m, F_m and G_m respectively.

By substituting these matrices into Equation 29 we have

$$C_0 = \mathbb{C}_1 = \mathbb{C}_2 = \mathbb{C}_3 = \mathbb{C}_4 = \mathbb{C}_5 = \mathbb{C}_6 = \mathbb{C}_7 = \bar{0},$$

where $\bar{\mathbf{0}}$ is the zero vector which can be written as $\bar{0} = [0,0,0,0]^T$.

For $p = 8$, it is found that $\mathbb{C}_8 \neq \bar{\mathbf{0}}$. Hence, the two-point implicit block method has order $p = 6$ with error constant $\mathbb{C}_8 = [\frac{1}{9450}, \frac{1}{17280}, \frac{1}{9450}, \frac{29}{604800}]^T$. For the three-point implicit block method, given by Equation 22, 23, 24, 25, 26 and 27, the formulae can be written in the form of a matrix as in Equation 30, where α, β, γ and δ are matrices (6x6) and

$$\alpha = \begin{bmatrix}0&0&0&0&0&0\\0&0&-1&1&0&0\\0&0&0&0&0&0\\0&0&0&-1&1&0\\0&0&0&0&0&0\\0&0&0&0&-1&1\end{bmatrix}, \beta = \begin{bmatrix}0&0&1&-1&0&0\\0&0&1&0&0&0\\0&0&0&1&-1&0\\0&0&0&1&0&0\\0&0&0&0&1&-1\\0&0&0&0&1&0\end{bmatrix},$$

$$\gamma = \begin{bmatrix}0&0&\frac{3463}{6480}&\frac{3537}{6480}&\frac{-783}{6480}&\frac{263}{6480}\\ 0&0&\frac{16384}{45360}&\frac{7857}{45360}&\frac{-2376}{45360}&\frac{815}{45360}\\ 0&0&\frac{-7}{80}&\frac{47}{80}&\frac{47}{80}&\frac{-7}{80}\\ 0&0&\frac{-715}{15120}&\frac{5832}{15120}&\frac{3051}{15120}&\frac{-608}{15120}\\ 0&0&\frac{263}{6480}&\frac{-783}{6480}&\frac{3537}{6480}&\frac{3463}{6480}\\ 0&0&\frac{38}{1680}&\frac{-115}{1680}&\frac{626}{1680}&\frac{291}{1680}\end{bmatrix}, \delta = \begin{bmatrix}0&0&\frac{97}{1080}&0&0&\frac{-17}{1080}\\ 0&0&\frac{388}{7560}&0&0&\frac{-53}{7560}\\ 0&0&\frac{-11}{360}&0&0&\frac{11}{360}\\ 0&0&\frac{-41}{2520}&0&0&\frac{36}{2520}\\ 0&0&\frac{17}{1080}&0&0&\frac{-97}{1080}\\ 0&0&\frac{22}{2520}&0&0&\frac{-97}{2520}\end{bmatrix},$$

$$Y_m = \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \\ y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix}, Y'_m = \begin{bmatrix} y'_{n-2} \\ y'_{n-1} \\ y'_n \\ y'_{n+1} \\ y'_{n+2} \\ y'_{n+3} \end{bmatrix}, F_m = \begin{bmatrix} f_{n-2} \\ f_{n-1} \\ f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix}, G_m = \begin{bmatrix} g_{n-2} \\ g_{n-1} \\ g_n \\ g_{n+1} \\ g_{n+2} \\ g_{n+3} \end{bmatrix}.$$

By substituting these matrices into (29) we have, $\mathbb{C}_0 = \mathbb{C}_1 = \cdots \mathbb{C}_6 = \mathbb{C}_7 = \bar{0}$,

It is found that, the three-point implicit block method has order $p = 6$ and error constant is $\mathbb{C}_8 = [\frac{97}{100800}, \frac{269}{604800}, \frac{-113}{100800}, \frac{-113}{201600}, \frac{97}{100800}, \frac{313}{604800}]^T$.

Zero-Stability of the Methods

For the two-point implicit block method, substituting Equation 11 into Equation 19, we have Equation 31

$$y'_{n+2} = y'_n + \frac{h}{15}[7f_n + 16f_{n+1} + 7f_{n+2} + \frac{h^2}{15}[g_n - g_{n+2}] \quad (31)$$

And also by substituting Equation 11 and 12 into Equation 20, we have Equation 32

$$y_{n+2} = y_n + 2hy'_n + h^2[\frac{79}{105}f_n + \frac{16}{15}f_{n+1} + \frac{19}{105}f_{n+2}] + h^3[\frac{2}{21}g_n - \frac{16}{105}g_{n+1} - \frac{4}{105}g_{n+2}]. \quad (32)$$

The first characteristic polynomial of the two-point implicit block method is given by,

$\rho(R) = \det[RA^{(0)} - A^{(1)}] = 0$, where

$$A^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } A^{(1)} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\rho(R) = \det \begin{bmatrix} R & 0 & -1 & 0 \\ 0 & R & 0 & -1 \\ 0 & 0 & R-1 & 0 \\ 0 & 0 & 0 & R-1 \end{bmatrix} = 0,$$

$R^2(R-1)^2 = 0, R = 0,0,1,1, |R| \leq 1.$

For the three-point implicit block method, substituting Equation 22 into Equation 24, to obtain Equation 33

$$y'_{n+2} = y'_n + h\left[\frac{181}{405}f_n + \frac{17}{15}f_{n+1} + \frac{7}{15}f_{n+2} - \frac{19}{405}f_{n+3}\right] + \frac{h^2}{135}[8g_n + 2g_{n+3}]. \quad (33)$$

Substituting Equation 22 and 23 into Equation 25, we obtain Equation 34

$$y_{n+2} = y_n + 2hy'_n + h^2[\frac{481}{567}f_n + \frac{116}{105}f_{n+1} + \frac{1}{35}f_{n+2} + \frac{52}{2835}f_{n+3}] + \frac{h^3}{945}[118g_n - 8g_{n+3}]. \quad (34)$$

Substituting Equation 33 into Equation 26, we have Equation 35

$$y'_{n+3} = y'_n + h\left[\frac{39}{80}f_n + \frac{81}{80}f_{n+1} + \frac{81}{80}f_{n+2} + \frac{39}{80}f_{n+3}\right] + \frac{h^2}{40}[3g_n - 3g_{n+3}] \quad (35)$$

And also by substituting Equation 33 and 34 into Equation 27, we have Equation 36

$$y_{n+3} = y_n + 3hy'_n + h^2[\frac{369}{280}f_n + \frac{243}{112}f_{n+1} + \frac{243}{280}f_{n+2} + \frac{81}{560}f_{n+3}] + h^3[\frac{27}{140}g_n - \frac{9}{280}g_{n+3}]. \quad (36)$$

The first characteristic polynomial of the three-point implicit block method is given as $\rho(R) = \det[RA^{(0)} - A^{(1)}] = 0,$

where

$$A^{(0)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } A^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\rho(R) = \det\begin{bmatrix} R & 0 & 0 & 0 & -1 & 0 \\ 0 & R & 0 & 0 & 0 & -1 \\ 0 & 0 & R & 0 & -1 & 0 \\ 0 & 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & 0 & R-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & R-1 \end{bmatrix} = 0, \; R^6 - 2R^5 + R^4 = 0,$$

$R = 0,0,0,0,1,1, |R| \leq 1.$

According to Ackleh et al. (2009), the two-point and three-point implicit block methods are zero-stable, since, the characteristic polynomial $\rho(\xi)$ has a modulus less than or equal to one, and that the multiplicity of the roots with modulus one be at most two.

RESULTS AND DISCUSSION

In this section, based on the new methods, codes in C-programming language are developed for solving general second order ordinary differential equation problems and the numerical results are compared when the same set of problems are solved using the existing methods. The comparisons are made with block methods of almost the same order and the same or higher step number. The values of $y'_{n+1}, y_{n+1}, y'_{n+2}, y_{n+2}$ in the two-point method and $y'_{n+1}, y_{n+1}, y'_{n+2}, y_{n+2}, y'_{n+3}$ and y_{n+3} in the three-point method are approximated using the predictor-corrector equations. Where Taylor method is used as the predictor equation, this is the same as in the implementation of other implicit block methods in the literature, see Majid et al. (2006) for further details. We are also using Taylor method for the predictor equations in the implementation of the comparison methods, hence it is a very fair comparison. The predictor equations using Taylor method for the two point method can be written as Equation 37,

$$y'^{p}_{n+m} = y^{p}_{n+(m-1)} + h\, f^{c}_{n+(m-1)},$$

$$y^{p}_{n+m} = y^{p}_{n+(m-1)} + h\, y'^{p}_{n+(m-1)} + \frac{h^2}{2!}\, f^{c}_{n+(m-1)}, \tag{37}$$

$$f^{p}_{n+m} = f(t_{n+m}, y^{p}_{n+m}, y'^{p}_{n+m}),$$

$$g^{p}_{n+m} = f'(t_{n+m}, y^{p}_{n+m}, y'^{p}_{n+m}).\ m = 1,2.$$

Problem 1 :

$y'' = 2y - y'.\ y(0) = 0,\ \ y'(0) = 1,\ \ [0,10].$

Exact Solution: $y(t) = \frac{e^{t} - e^{-2t}}{3}.$

Problem 2 :

$t^2 y'' + t y' + (t^2 - 0.25)y = 0.\ y(1) = \sqrt{\frac{2}{\pi}} sin1,\ \ y'(1) = \frac{2cos1 - sin1}{\sqrt{2\pi}},\ \ [1,8].$

Exact Solution: $y(t) = \sqrt{\frac{2}{\pi t}} sin(t).$

Problem 3:

$y'' - t(y')^2 = 0.\ y(0) = 1,\ \ y'(0) = \frac{1}{2},\ \ [0,1].$

Exact Solution: $y(t) = 1 + \frac{1}{2} ln(\frac{2+t}{2-t}).$

Problem 4 :

$y''_1 = -y_2 + sin\pi t, \qquad y_1(0) = 0, \; y'_1(0) = -1,$

$y''_2 = -y_1 + 1 - \pi^2 sin\pi t, \quad y_2(0) = 1, \; y'_2(0) = 1 + \pi, \; [0,5].$

Exact Solution: $y_1(t) = 1 - e^t, y_2(t) = e^t + sin\pi t.$

Problem 5:

$y''_1 = \frac{-y_1}{r^3}, \qquad y_1(0) = 1, \; y'_1(0) = 0,$

$y''_2 = \frac{-y_2}{r^3}, \quad y_2(0) = 0, \; y'_2(0) = 1, \; r = \sqrt{y_1^2 + y_2^2}, \; [0,10].$

Exact Solution: $y_1(t) = cos(t), y_2(t) = sin(t).$

Problem 6:

$y'' = 100y, \; y(0) = 1, \; y'(0) = -10, \; [0,2].$

Exact Solution: $y(t) = e^{-10t}.$

From the set of test problems, problems 1, 2 and 4 are linear problems. Problems 3 and 5 are nonlinear problems and problem 6 is a mildly stiff problem. Problem 5 is also the two body problem which determines the motion of two objects interact with each other.

Notations used are:

h : step size.
Time : time in seconds.
Max Error : maximum error $|y(t_i) - y_i|$.
2PSDBI(2) : New two-point implicit second derivative block method of order six.
3PSDBI(2) : New three-point implicit second derivative block method of order six.
Majid(2) : Order three, Two-point implicit block method in Majid et al. (2012).
Omar : Order five, Implicit Four-point block method in Omar and Adeyeye (2016)
Awoyemi(2): Order four, Implicit Three-point modified block method in Awoyemi et al. (2011).
Mukhtar(2) : Four- point implicit block method in Mokhtar et al. (2012).

1.2345(-6) means $1.2345\text{x}10^{-6}$. Numerical results for 2PSDBI(2) are given in Figure 1 to 6, whereas for 3PSDBI(2) are given in Figure 7 to 12 respectively.

For methods with less algebraic order usually the accuracy is less but the total computational time is also less since it has less function evaluations or less number of steps in the formula. For method with higher algebraic order the accuracy is more but the computational time is also more because there are more steps and more function

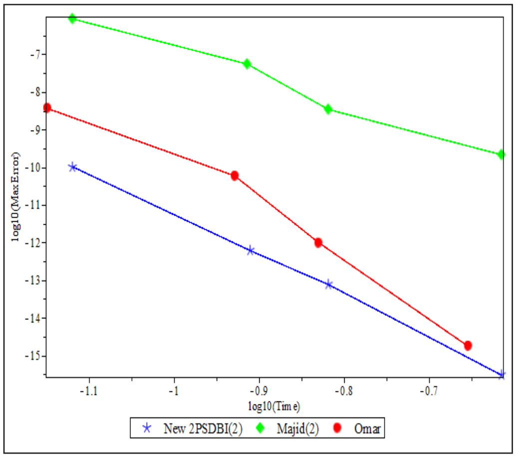

Figure 1. Efficiency curves (2PSDBI(2)) for Problem 1

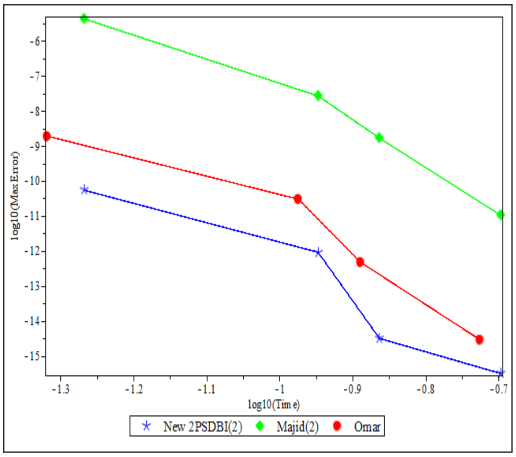

Figure 2. Efficiency curves (2PSDBI(2)) for Problem 2

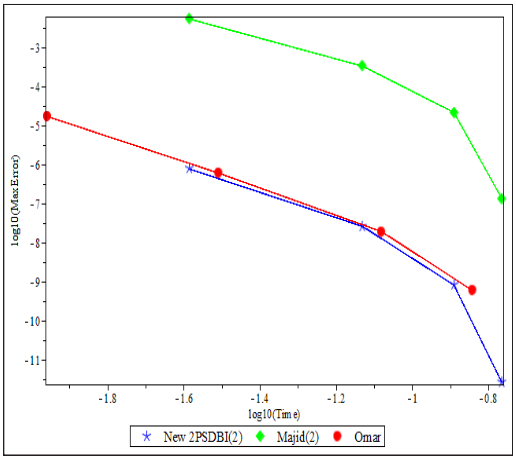

Figure 3. Efficiency curves (2PSDBI(2)) for Problem 3

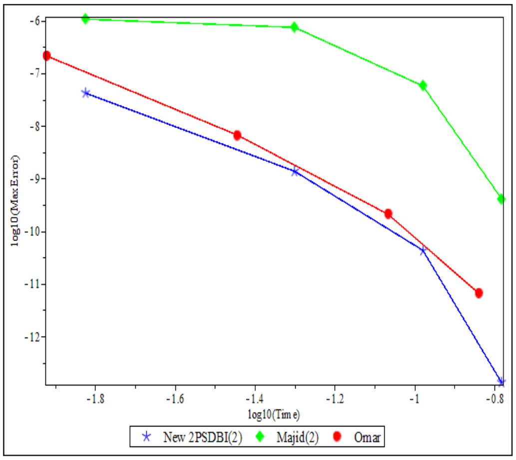

Figure 4. Efficiency curves (2PSDBI(2)) for Problem 4

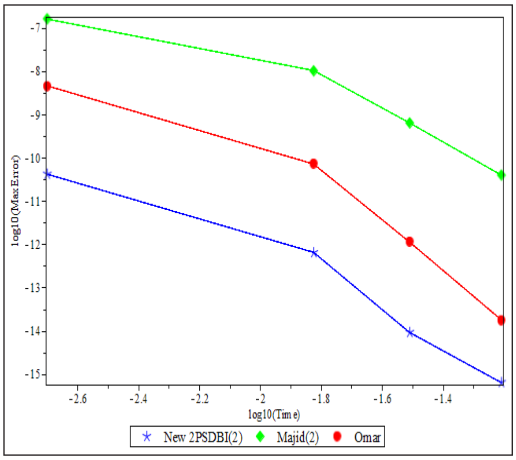

Figure 5. Efficiency curves (2PSDBI(2)) for Problem 5

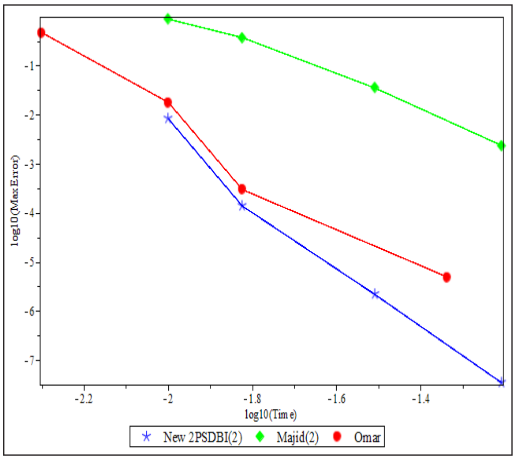

Figure 6. Efficiency curves (2PSDBI(2)) for Problem 6

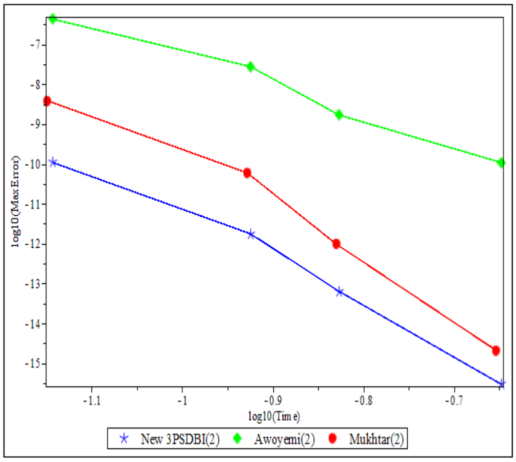

Figure 7. Efficiency curves (3PSDBI(2)) for Problem 1

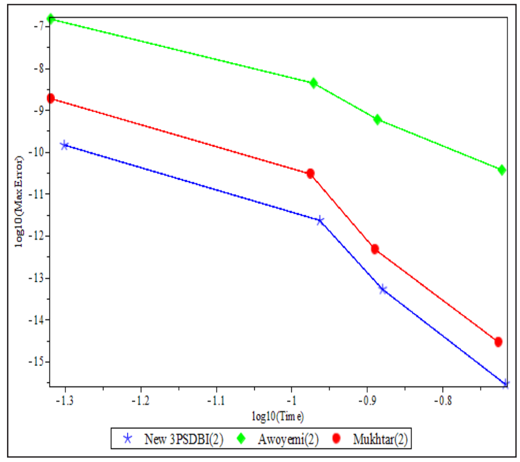

Figure 8. Efficiency curves (3PSDBI(2)) for Problem 2

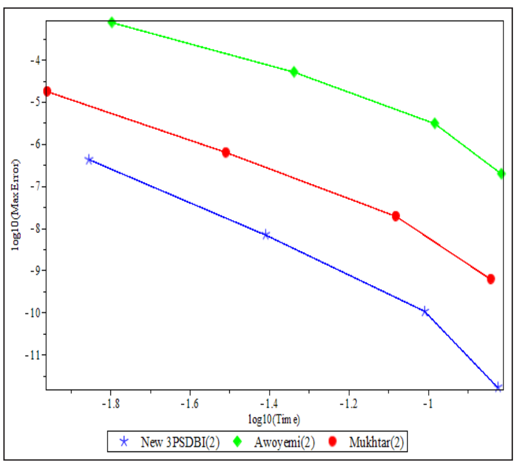

Figure 9. Efficiency curves (3PSDBI(2)) for Problem 3

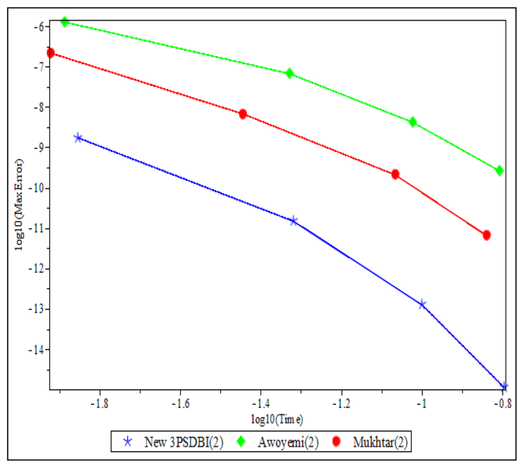

Figure 10. Efficiency curves (3PSDBI(2)) for Problem 4

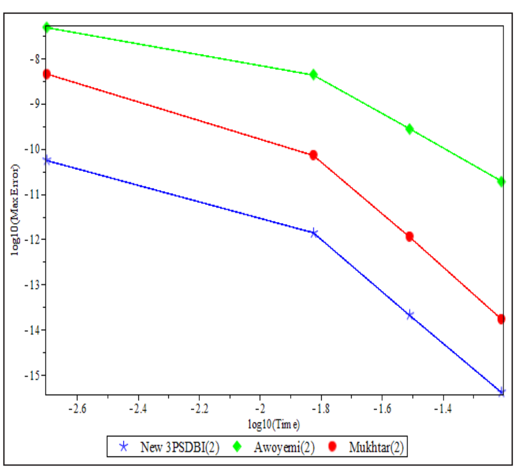

Figure 11. Efficiency curves (3PSDBI(2)) for Problem 5

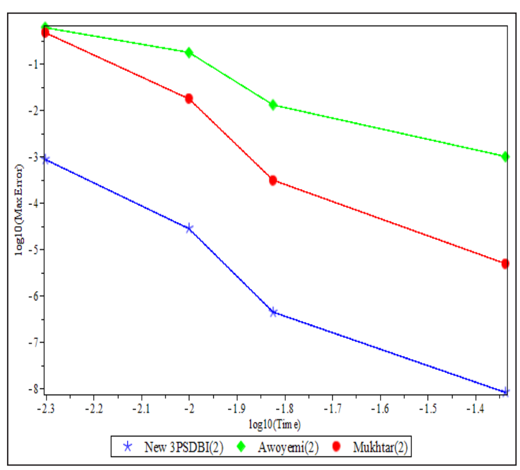

Figure 12. Efficiency curves (3PSDBI(2)) for Problem 6

evaluations in the formula. Thus, the right technique of measuring the efficiency of certain numerical methods is by using the efficiency curves. Figure1 to 12, showed the efficiency curves, where the common logarithm of the maximum global errors were plotted versus the computational time. From the efficiency curves given in Figure 1 to 6, it is observed that 2PSDBI(2) method is the most efficient compared to Majid(2) and Omar for solving the same set of test problems, since a smaller global maximum error can be attained for the same total of computational time. The same observation can be seen in Figure 7 to 12, it is obvious that the new 3PSDBI(2) method performed better than Awoyemi(2) and Mukhtar(2) methods.

CONCLUSION

We presented the construction of two and three-point extra derivative implicit block methods for directly solving general second order IVPs. The order and zero-stability of the methods are given. The methods are then used to solve linear, nonlinear and mildly stiff IVPs. From the efficiency curves, we can be conclude that the proposed methods performed noticeably more efficient than the existing methods, though the methods of comparisons are of the same nature as the proposed methods, that is block in nature and can directly solve general second order IVPs. Therefore, the proposed methods have a very high potential to be an efficient numerical methods for integrating general second order IVPs.

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